Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 233–242, 1996

0021–8928/96 $24.00+0.00

S0021–8928(96)00030-5

THE OSCILLATIONS OF A PARTICLE SUSPENDED ON AN IDEAL THREAD†

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(*Received* 22 *March* 1995)

The motion of a particle suspended on an ideal thread in a uniform gravitational field is investigated. The problem of the orbital stability of the periodic motion of the particle along the vertical is solved. The non-linear oscillations in the neighbourhood of the periodic motion are considered in the case when the motion is unstable. Normalization of the Hamilton function using simplectic mappings is employed in the investigation.

1. FORMULATION OF THE PROBLEM

Suppose a particle of mass m is attached at one of the ends of an absolutely flexible weightless inextensible thread of length l, the other end of which is attached to a fixed point O. The particle moves in a uniform gravitational field in a fixed vertical plane Oxy (Fig. 1).

Suppose x, y are the coordinates of the particle m. The condition for the thread to be inextensible gives the unilateral constraint $l^2 - x^2 - y^2 \geqslant 0$. We will take as generalized coordinates the polar angle θ and the quantity $\xi = l - (x^2 + y^2)^{1/2}$. The corresponding momenta will be the quantities

$$p_\theta = m(l-\xi)^2\dot{\theta}, \quad p_\xi = m\dot{\xi}$$

In the case of a weakened relationship, when $\xi > 0$, the motion is described by canonical equations with the Hamilton function

$$H = \frac{1}{2m}\left[p_\xi^2 + \frac{p_\theta^2}{(l-\xi)^2}\right] - mg(l-\xi)\cos\theta \tag{1.1}$$

If, at the instant when the relation $\xi = 0$ is attained the value of p_ξ is non-zero, a shock occurs. The following relation is satisfied for the shock

$$p_\xi^+ = -p_\xi^- \tag{1.2}$$

where the minus and plus superscripts denote the values of the momentum p_ξ before and after the shock.

The equations of motion have a particular solution corresponding to motion of the particle m along the fixed vertical Ox (Fig. 1). The period τ is equal to $2(2h/g)^{1/2}$, where g is the acceleration due to gravity and h is the height that the particle jumps along the vertical when the thread is loose after the shock. To eliminate collision between the particle m and the point O of the fixed thread we will assume that $h < l$.

This τ-periodic motion in the time interval $0 \leqslant t < \tau$ is described by the equations

$$\begin{gathered}\theta = 0, \ p_\theta = 0 \\ \xi = -\tfrac{1}{2}gt^2 + (2gh)^{1/2}t, \quad p_\xi = -mgt + m(2gh)^{1/2}\end{gathered} \tag{1.3}$$

The functions $\xi(t), p_\xi(t)$, when $\xi > 0$, satisfy the canonical equations with Hamiltonian

$$\Gamma = \frac{1}{2m}p_\xi^2 + mg\xi \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 240–249, 1996.

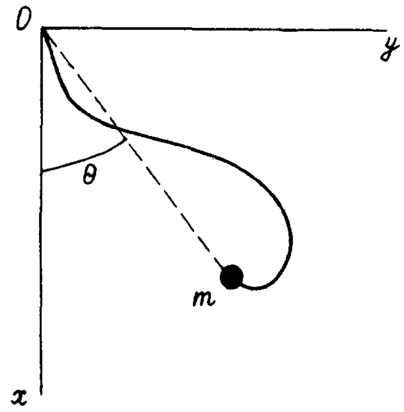

Fig. 1.

and, when $\xi = 0$, Eq. (1.2) is satisfied. For shocks which occur at instants of time that are a multiple of τ, we have $\xi = 0$, while p_ξ jumps in value from $p_\xi^- = -m(2gh)^{1/2}$ to $p_\xi^+ = -m(2gh)^{1/2}$.

The purpose of the present paper is to give a strict analytic solution of the non-linear problem of the orbital stability of the periodic motion (1.3) for all values of h in the interval $0 < h < l$. We also investigate the non-linear oscillations of a particle in the neighbourhood of the motion (1.3) when the value of h lies in the range of its orbital instability. The stability of the motion (1.3) was considered previously in [1–3] in the linear form of the problem. A number of other types of periodic motions, differing from (1.3), were investigated in [2, 3], and the phenomenon of chaotic motion was studied.

2. THE HAMILTONIAN OF THE PERTURBED MOTION

Following [4], in (1.1) we make a canonical change of variables, keeping the quantities θ, p_θ unchanged and introducing quantities J and v instead of the variables p_ξ and ξ, as given by the following formulae

$$p_\xi = (3m^2\pi g)^{1/3} J^{1/3} f_1(v), \quad \xi = \left(\frac{9\pi^2}{8m^2 g}\right)^{1/3} J^{2/3} f_2(v) \tag{2.1}$$

where f_1 and f_2 are 2π-periodic functions of v, and we have $f_1 = 1 - v\pi^{-1}$, $f_2 = v\pi^{-1}(2 - v\pi^{-1})$ for $0 \leqslant v < 2\pi$.

In the unperturbed motion with Hamiltonian (1.4) the variables J and v will be the action-angle variables I and w. Here

$$\Gamma = \left(\frac{9m\pi^2 g^2}{8}\right)^{1/3} I^{2/3}, \quad I = \frac{2m(2g)^{1/2}}{3\pi} h^{3/2} \tag{2.2}$$

$$\frac{\partial \Gamma}{\partial I} = \frac{2\pi}{\tau}, \quad w = \frac{2\pi}{\tau} t = \pi\sqrt{\frac{g}{2h}}\, t$$

To obtain the Hamilton function, which describes the motion in the neighbourhood of the periodic motion (1.3), we introduce the perturbations q, p and r using the canonical transformation $\theta, p_\theta, J, v \to q, p, r, v$, given by the equations

$$\theta = q, \quad p_\theta = \frac{ml^2}{\pi}\sqrt{\frac{2g}{h}}\, p, \quad J = I + \frac{ml^2}{\pi}\sqrt{\frac{2g}{h}}\, r, \quad v = v \tag{2.3}$$

If we change from t to the new dimensionless independent variable $\pi(g/2h)^{1/2}t$, we obtain from (1.1) and (2.1)–(2.3) the Hamiltonian of the perturbed motion in the form of a series in even powers of q, p, $|r|^{1/2}$

$$H = r + h_2(q, p, \nu) + \dots \tag{2.4}$$

$$h_2 = \tfrac{1}{2}\varkappa(1 - \varkappa f_2)q^2 + \pi^{-2}(1 - \varkappa f_2)^{-2}p^2, \quad \varkappa = h/l \quad (0 < \varkappa < 1)$$

where the dots denote terms of the fourth and higher powers. The term in (2.4) that is independent of q, p, r, ν is omitted.

The orbital stability (instability) of periodic motion (1.3) denotes the stability (instability) of the solution $q = p = r = 0$ of the system with Hamiltonian (2.4) with respect to perturbations of the quantities q, p, r.

3. THE NORMALIZATION METHOD

Suppose the perturbed motion occurs at the same energy level as the unperturbed periodic motion (1.3). Then $H = 0$, where H is the function (2.4). From the equation $H = 0$ we obtain $r = -K(q, p, \nu)$. At the isoenergy level considered the perturbed motion is described by canonical equations with Hamiltonian K (Whittaker's equations [5]), and the quantity ν plays the role of the independent variable. The solution $q = p = 0$ of these equations corresponds to the periodic motion (1.3) being investigated. The orbital stability (instability) of the periodic motion (1.3) follows from the stability (instability) of the solution $q = p = 0$.

In the neighbourhood of the point $q = p = 0$, the function K can be represented in the form of a converging series in powers of q and p

$$K = K_2 + K_4 + \dots + K_s + \dots \tag{3.1}$$

where K_s is a form of power s with coefficients that are 2π-periodic in ν, and $K_2 = h_2$. Using existing algorithms [6], when solving the problems of the stability of the equilibrium $q = p = 0$ it is necessary to normalize the first few forms of the expansion (3.1), which is an extremely tedious procedure. In particular, when normalizing the quadratic form K_2 one has to find (either numerically or analytically) the fundamental matrix of the solutions of the corresponding linear system of differential equations, 2π-periodic in ν. It is true that in the problem considered this matrix can be written down in explicit forms, but for a known matrix of the fundamental solutions the normalization of forms higher than the second power in (3.1) requires the evaluation of several definite integrals in the interval $0 \leqslant \nu \leqslant 2\pi$. This can turn out to be extremely complicated, particularly if it is necessary to obtain explicit expressions for the coefficients of the normal form in terms of the parameters of the problem.

We will use a different method of normalization in this paper. It is based on an investigation of the non-linear simplectic mapping, specified by the motions of the system with Hamiltonian K during the period when ν varies from 0 to 2π. The basic principle of the method is briefly as follows. Suppose q_0 and p_0 are the initial value of the variables q and p when $\nu = 0$ or, which is the same thing, when $t = 0$, and q_1 and p_1 are their values when $\nu = 2\pi$ (or, which is the same thing, when $t = t_1$, where t_1 is the instant when the particle satisfies the relation $x^2 + y^2 = l^2$). The functions $q_1 = q_1(q_0, p_0)$, $p_1 = p_1(q_0, p_0)$ specify the simplectic mapping $q_0, p_0 \to q_1, p_1$. This mapping has a fixed point $q_0 = p_0 = 0$, while the functions q_1 and p_1 are analytic in the neighbourhood of this point. Using a canonical transformation this mapping can be reduced to a normal form, from the form of which one can then obtain the corresponding normal form of the Hamiltonian K.

A similar approach to the problem of normalizing time-periodic Hamiltonian systems was proposed in [7], where the simplectic mapping was obtained using numerical integration of the corresponding canonical equations using Jacobi's method. In the problem investigated here, the mapping $q_0, p_0 \to q_1, p_1$ is obtained without numerical integration. The procedure for constructing it uses the fact that when ν varies from 0 to 2π, the thread on which the particle m is suspended becomes loose and the motion of the point is completely known. It is given by the equations

$$x(t) = \tfrac{1}{2}gt^2 + \dot{x}_0 t + x_0, \quad y(t) = \dot{y}_0 t + y_0 \tag{3.2}$$

$$\dot{x}(t) = gt + \dot{x}_0, \quad \dot{y}(t) = \dot{y}_0$$

where $x_0, y_0, \dot{x}_0, \dot{y}_0$ are the values of the corresponding quantities when $\nu = 0$.

4. THE MAPPING

Using the equation $r_0 = -K(q_0, p_0, 0)$, the formulae connecting the initial variables $\theta, p_\theta, \xi, p_\xi$ in Sections 1 and 2, and the variables q, p, r and v, and also the relation between $x, y, \dot{x}, \dot{y}$ and $\theta, p_\theta, \xi, p_\xi$, not written here, we obtain the following expressions for the quantities $x_0, y_0, \dot{x}_0, \dot{y}_0$ in terms of q_0 and p_0

$$x_0 = l(1 - \tfrac{1}{2}q_0^2 + O_4), \quad y_0 = l(q_0 - \tfrac{1}{6}q_0^3 + O_4) \tag{4.1}$$

$$\dot{x}_0 = -\sqrt{2gh}\left(1 - \frac{2\varkappa+1}{4\varkappa}q_0^2 + \frac{1}{\pi\varkappa}q_0p_0 - \frac{1}{2\pi^2\varkappa^2}p_0^2 + O_4\right)$$

$$\dot{y}_0 = -\sqrt{2gh}\left(q_0 - \frac{1}{\pi\varkappa}p_0 - \frac{2\varkappa+3}{12\varkappa}q_0^3 + \frac{1}{2\pi\varkappa}q_0^2p_0 - \frac{1}{2\pi^2\varkappa^2}q_0p_0^2 + O_4\right)$$

where O_4 is the set of terms higher than the third power in q_0 and p_0.

The relation $x^2 + y^2 = l^2$, taking (3.2) and (4.1) into account, gives an equation for finding the instant of time t_1 when the particle m reaches the constraint. Solving it we obtain

$$t_1 = 2\sqrt{\frac{2h}{g}}\left(1 - \frac{8\varkappa^2 - 2\varkappa + 1}{4\varkappa}q_0^2 + \frac{4}{\pi}q_0p_0 - \frac{4\varkappa+1}{2\pi^2\varkappa^2}p_0^2 + O_4\right)$$

Putting $t = t_1$ in (3.2) and taking (4.1) into account we obtain the values $x_1, y_1, \dot{x}_1, \dot{y}_1$ of the quantities $x, y, \dot{x}, \dot{y}$ at the instant of reaching the constraint, expressed in terms of q_0 and p_0. Then taking the chain of replacements of variables in the reverse order, carried out when obtaining relations (4.1), we find the following relations, which express the required mapping of the expression q_1, p_1 of the variables q, p when $v = 2\pi$ in terms of their values when $v = 0$

$$q_1 = (1 - 4\varkappa)q_0 + 4/\pi p_0 + a_{30}q_0^3 + a_{21}q_0^2p_0 + a_{12}q_0p_0^2 + a_{03}p_0^3 + O_4 \tag{4.2}$$

$$p_1 = -2\pi\varkappa(1 - 2\varkappa)q_0 + (1 - 4\varkappa)p_0 + b_{30}q_0^3 + b_{21}q_0^2p_0 + b_{12}q_0p_0^2 + b_{03}p_0^3 + O_4$$

$$\begin{aligned} a_{30} &= -\tfrac{2}{3}(16\varkappa^3 - 24\varkappa^2 + 5\varkappa - 3), \quad a_{21} = \varkappa^{-1}\pi^{-1}(32\varkappa^3 - 40\varkappa^2 + 2\varkappa - 1) \\ a_{12} &= -4\varkappa^{-1}\pi^{-2}(8\varkappa^2 - 8\varkappa - 1), \quad a_{03} = \tfrac{2}{3}\varkappa^{-2}\pi^{-3}(16\varkappa^2 - 12\varkappa - 3) \\ b_{30} &= -\tfrac{1}{6}\pi(96\varkappa^3 - 44\varkappa^2 + 22\varkappa - 3), \quad b_{21} = 2(24\varkappa^2 - 5\varkappa + 1) \\ b_{12} &= -\varkappa^{-1}\pi^{-1}(48\varkappa^2 + 2\varkappa - 1), \quad b_{03} = 4\varkappa^{-1}\pi^{-2}(4\varkappa + 1) \end{aligned} \tag{4.3}$$

5. STABILITY IN THE LINEAR APPROXIMATION

The fundamental matrix of the linearized equations of the perturbed motion with Hamiltonian K_2, calculated for $v = 2\pi$, is identical with the matrix of the linearized mapping (4.2). Its characteristic equation has the form

$$\rho^2 - 2(1 - 4\varkappa)\rho + 1 = 0 \tag{5.1}$$

When the inequality

$$0 < \varkappa < \tfrac{1}{2} \tag{5.2}$$

is satisfied, the roots (multipliers) of Eq. (5.1) are complex conjugate numbers with moduli equal to unity: $\rho_1 = \exp(i2\pi\lambda)$, $\rho_2 = \rho_1$ where $\pm i\lambda$ are the characteristic exponents of the linear equations with Hamiltonian K_2. In the range given by (5.2) the motion (1.3) is orbitally stable in the linear approximation [8].

At the boundary $\varkappa = 1/2$ of the range (5.2) the multipliers are multiples: $\rho_1 = \rho_2 = -1$. Here there is instability in the linear approximation since the matrix of the linearized mapping (4.2) is not reduced to diagonal form.

When the inequality $\varkappa > 1/2$ is satisfied, Eq. (5.1) has a root with modulus greater than unity and, consequently [8], there is instability (in the rigorous non-linear formulation of the problem, and not only in its linear approximation). This indicates that if, in the unperturbed periodic motion (1.3), the height of the post-impact jump of the particle m exceeds half the length of the thread, the periodic motion considered is orbitally unstable.

When $0 < \varkappa \leqslant 1/2$ we have the critical case of the theory of stability and a non-linear analysis is required for a rigorous solution of the problem of the orbital stability of the motion (1.3).

6. NON-LINEAR ANALYSIS IN THE STABILITY REGION IN THE FIRST APPROXIMATION

Suppose $\varkappa$ lies inside the range (5.2). It then follows from (5.1) that $\cos 2\pi\lambda = 1 - 4\varkappa$. Hence (taking into account the fact that for $\varkappa = 0$ we have $K_2 = \pi^{-2}p^2$ and, consequently, as $\varkappa \to 0$ the quantity $\lambda \to 0$), we obtain that in the range (5.2)

$$\lambda = (2\pi)^{-1}\arccos(1 - 4\varkappa) \tag{6.1}$$

By making the change of variables $q = \mu^{-1}q'$, $p = \mu p'$, where $\mu = 1/2(\pi \sin 2\pi\lambda)^{1/2}$ we reduce the linear part of the mapping (4.2) to the normal form—rotation by an angle $2\pi\lambda$. The mapping takes the following form (the primes on the new variables are omitted)

$$\begin{aligned} q_1 &= \cos 2\pi\lambda q_0 + \sin 2\pi\lambda p_0 + c_{30}q_0^3 + c_{21}q_0^2 p_0 + c_{12}q_0 p_0^2 + c_{03}p_0^3 + O_4 \\ p_1 &= -\sin 2\pi\lambda q_0 + \cos 2\pi\lambda p_0 + d_{30}q_0^3 + d_{21}q_0^2 p_0 + d_{12}q_0 p_0^2 + d_{03}p_0^3 + O_4 \end{aligned} \tag{6.2}$$

$$\begin{aligned} &c_{30} = \mu^{-2}a_{30}, \quad c_{21} = a_{21}, \quad c_{12} = \mu^2 a_{12}, \quad c_{03} = \mu^4 a_{03} \\ &d_{30} = \mu^{-4}b_{30}, \quad d_{21} = \mu^{-2}b_{21}, \quad d_{12} = b_{12}, \quad d_{03} = \mu^2 b_{03} \end{aligned} \tag{6.3}$$

The coefficients c_{kl} and d_{kl} are related by the identities

$$\begin{aligned} &\cos 2\pi\lambda(3c_{30} + d_{21}) = \sin 2\pi\lambda(3d_{30} - c_{21}), \quad \cos 2\pi\lambda(c_{21} + d_{12}) = \sin 2\pi\lambda(d_{21} - c_{12}) \\ &\cos 2\pi\lambda(c_{12} + 3d_{03}) = \sin 2\pi\lambda(d_{12} - 3c_{03}) \end{aligned} \tag{6.4}$$

These identities are a consequence of the simplectic form of the mapping (4.2).

We will now reduce the terms of the third power in the mapping (6.2) to normal form. To do this it is first convenient to change from q and p to the complex-conjugate variables $z, \bar{z}$: $z = q - ip$, $\bar{z} = q + ip$. The mapping can be written in the variables $z, \bar{z}$ in the following form

$$\begin{aligned} z_1 &= \rho_1 z_0 + f_{30}z_0^3 + f_{21}z_0^2\bar{z}_0 + f_{12}z_0\bar{z}_0^2 + f_{03}\bar{z}_0^3 + O_4 \\ \bar{z}_1 &= \rho_2 \bar{z}_0 + g_{30}z_0^3 + g_{21}z_0^2\bar{z}_0 + g_{12}z_0\bar{z}_0^2 + g_{03}\bar{z}_0^3 + O_4 \\ f_{kl} &= \mu_{kl} + i\nu_{kl}, \quad g_{kl} = \bar{f}_{lk} \end{aligned} \tag{6.5}$$

$$\begin{aligned} &\mu_{30} = \tfrac{1}{8}(c_{30} - c_{12} + d_{21} - d_{03}), \quad \nu_{30} = -\tfrac{1}{8}(d_{30} - c_{21} - d_{12} + c_{03}) \\ &\mu_{21} = \tfrac{1}{8}(3c_{30} + c_{12} + d_{21} + 3d_{03}), \quad \nu_{21} = -\tfrac{1}{8}(3d_{30} - c_{21} + d_{12} - 3c_{03}) \\ &\mu_{12} = \tfrac{1}{8}(3c_{30} + c_{12} - d_{21} - 3d_{03}), \quad \nu_{12} = -\tfrac{1}{8}(3d_{30} + c_{21} + d_{12} + 3c_{03}) \\ &\mu_{03} = \tfrac{1}{8}(c_{30} - c_{12} - d_{21} + d_{03}), \quad \nu_{03} = -\tfrac{1}{8}(d_{30} + c_{21} - d_{12} - c_{03}) \end{aligned} \tag{6.6}$$

We make the canonical transformation $z, \bar{z} \to z^*, \bar{z}^*$ specified implicitly by the equations

$$z^* = \partial S / \partial \bar{z}^*, \quad \bar{z} = \partial S / \partial z$$

where

$$S = z\bar{z}^* + s_{40}z^4 + s_{31}z^3\bar{z}^* + s_{22}z^2\bar{z}^{*2} + s_{13}z\bar{z}^{*3} + s_{04}\bar{z}^{*4}$$

and the coefficients s_{mn} are chosen so as to simplify the structure of the mapping to the greatest extent.

Mapping (6.5) takes the following form in the new variables

$$\begin{aligned}
z_1^* &= \rho_1 z_0^* + [f_{30} + \rho_1(\rho_1^2 - 1)s_{31}]z_0^{*3} + f_{21}z_0^{*2}\bar{z}_0^* + \\
&+ [f_{12} + 3\rho_1(\rho_2^2 - 1)s_{13}]z_0^*\bar{z}_0^{*2} + [f_{03} + 4\rho_1(\rho_2^4 - 1)s_{04}]\bar{z}_0^{*3} + O_4 \\
\bar{z}_1^* &= \rho_2\bar{z}_0^* + [g_{30} - 4\rho_2(\rho_1^4 - 1)s_{40}]z_0^{*3} + [g_{21} - 3\rho_2(\rho_1^2 - 1)s_{31}]z_0^{*2}\bar{z}_0^* + \\
&+ g_{12}z_0^*\bar{z}_0^{*2} + [g_{03} - \rho_2(\rho_2^2 - 1)s_{13}]\bar{z}_0^{*3} + O_4
\end{aligned} \tag{6.7}$$

For values of the parameter $\varkappa$ in the range (5.2) we have $\rho_1^2 \neq 1$, $\rho_2^2 \neq 1$. The quantities ρ_1^4 and ρ_2^4 differ from unity for all values of $\varkappa$ in the range (5.2) apart from $\varkappa = 1/4$, for which $\rho_1^4 = \rho_2^4 = 1$. For this value of $\varkappa$ we obtain the fourth-order resonance $4\lambda = 1$.

We will first consider the non-resonant case $\varkappa \neq 1/4$. Then the coefficients s_{mn} of the generating function S can be chosen so that only one third-degree monomial remains on the right-hand sides of each of Eqs (6.7), and the mapping takes the following (normal) form

$$z_1^* = \rho_1 z_0^* + f_{21}z_0^{*2}\bar{z}_0^* + O_4, \quad \bar{z}_1^* = \rho_2\bar{z}_0^* + g_{12}z_0^*\bar{z}_0^{*2} + O_4$$

This mapping corresponds to Hamilton function (3.1), normalized up to terms of the fourth power

$$K = i\lambda z^*\bar{z}^* + \tfrac{1}{2}ic_2(z^*\bar{z}^*)^2 + \ldots \tag{6.8}$$

where $c_2 = -i\rho_2(2\pi)^{-1}f_{21}$. It follows from (6.4) and (6.6) that c_2 is real and can be calculated from the formula

$$c_2 = -(3c_{30} + d_{21} + 3d_{03} + c_{12})/(8\mu)^2 \tag{6.9}$$

In the canonically conjugate real variables φ^* and R^*, introduced by the canonical representation

$$z^* = -i(2R^*)^{1/2}\exp(i\varphi^*), \quad \bar{z}^* = i(2R^*)^{1/2}\exp(-i\varphi^*) \tag{6.10}$$

Hamiltonian (6.8) has the form

$$K = \lambda R^* + c_2 R^{*2} + O(R^{*3}) \tag{6.11}$$

If the quantity c_2 in (6.11) is non-zero, the position of equilibrium $q = p = 0$ of the system with Hamilton function (3.1) is stable [9, 10]. Calculations using (6.9), (6.3) and (4.3) show that the expression for c_2 can be converted to the form

$$c_2 = -\frac{\varkappa + 4}{8\pi^2\varkappa(1 - 2\varkappa)}$$

In the range (5.2) we have $c_2 < 0$ and consequently when $\varkappa - 1/4$ in this range the periodic motion (1.3) is orbitally stable. Suppose now that $\varkappa = 1/4$. We then have resonance $4\lambda = 1$ and the normal form of the mapping will be

$$z_1^* = \rho_1 z_0^* + f_{21}z_0^{*2}\bar{z}_0^* + f_{03}\bar{z}_0^{*3} + O_4, \quad \bar{z}_1^* = \rho_2\bar{z}_0^* + g_{12}z_0^*\bar{z}_0^{*2} + g_{30}z_0^{*3} + O_4$$

For the corresponding Hamilton function (3.1), normalized up to terms of the fourth power, we obtain the expression

$$K = i\lambda z^*\bar{z}^* + \tfrac{1}{2}ic_2(z^*\bar{z}^*)^2 + (8\pi)^{-1}\rho_2 f_{03}e^{i\upsilon}\bar{z}^{*4} - (8\pi)^{-1}\rho_1 g_{30}e^{-i\upsilon}z^{*4} + \ldots$$

In φ^*, R^* variables, defined by (6.10), we have

$$K = \lambda R^* + c_2 R^{*^2} - \\ -(2\pi)^{-1}[\mu_{03}\sin(2\pi\lambda + 4\varphi^* - \upsilon) - \nu_{03}\cos(2\pi\lambda + 4\varphi^* - \upsilon)]R^{*^2} + O(R^{*^3}) \tag{6.12}$$

When $\varkappa = 1/4$ we have $c_2 = -17(2\pi)^{-2}$, $\mu_{03} = -(2\pi)^{-1}$, $\nu_{03} = 0$. In $R = R^*$, $\varphi = \varphi^* - \nu/4$ variables, instead of (6.12) we obtain the Hamiltonian

$$K = (c_2 + b_2\cos 4\varphi)R^2 + O(R^3)$$

where $b_2 = (2\pi)^{-2}$. Since $|c_2| > b_2$, in the resonance case considered the periodic motion (1.3) is orbitally stable [7].

7. STABILITY AT THE BOUNDARY OF THE RANGE (5.2)

When $\varkappa \neq 1/2$ the quantity λ is equal to 1/2, i.e. we obtain second-order resonance. In this case it is not possible to reduce the quadratic part of the Hamilton function (3.1) to a Hamiltonian, independent of ν, in the class of linear canonical transformations that are 2π-periodic in ν [11]. But this reduction can be achieved in the class of 4π– periodic transformations. According to this, when the function K is normalized, instead of the mapping $q_0, p_0 \to q_1, p_1$ during the period of variation of ν from 0 to 2π we will consider the mapping $q_0, p_0 \to q_2, p_2$ during twice the period of variation of ν from 0 to 4π. For $\varkappa = 1/2$ we obtain from (4.2)

$$\begin{aligned} &q_2 = q_0 - 8\pi^{-1}p_0 + F(q_0, p_0) + O_4, \quad p_2 = p_0 + G(q_0, p_0) + O_4 \\ &F = -12q_0^3 + 96\pi^{-1}q_0^2 p_0 - 384\pi^{-2}q_0 p_0^2 + 1808(3\pi^3)^{-1}p_0^3 \\ &G = 3\pi q_0^3 - 36q_0^2 p_0 + 192\pi^{-1}q_0 p_0^2 - 384\pi^{-2}p_0^3 \end{aligned} \tag{7.1}$$

We make the following canonical change of variables $q, p \to q^*, p^*$ specified by the equations

$$\begin{aligned} &q^* = \partial S / \partial p^*, \quad p = \partial S / \partial q \\ &S = (\pi\sqrt{2}/2)qp^*(1 - (11/60)p^{*^2}) \end{aligned}$$

In the q^*, p^* variables, mapping (7.1) takes the following form

$$\begin{aligned} &q_2^* = q_0^* - 4\pi p_0^* + F^*(q_0^*, p_0^*) + O_4, \quad p_2^* = p_0^* + G^*(q_0^*, p_0^*) + O_4 \\ &F^* = -24\pi^{-2}q_0^{*^3} + 96\pi^{-1}q_0^{*^2}p_0^* - 192q_0^* p_0^{*^2} + 768\pi/5\, p_0^{*^3} \\ &G^* = 12\pi^{-3}q_0^{*^3} - 72\pi^{-2}q_0^{*^2}p_0^* + 192\pi^{-1}q_0^* p_0^{*^2} - 192p_0^{*^3} \end{aligned} \tag{7.2}$$

This mapping is generated (in the range of variation of ν from 0 to 4π) by the Hamiltonian (3.1), normalized up to terms of the fourth power

$$K = -\tfrac{1}{2}p^{*^2} - \tfrac{3}{4}\pi^{-4}q^{*^4} + O_6 \tag{7.3}$$

Since the signs of the coefficients of p^{*2} and q^{*4} in (7.3) are the same, the equilibrium $q = p = 0$ in a system with Hamilton function (3.1) is stable [12].

The above investigation enables us to formulate the following final result: if in the unperturbed periodic motion of a point along the vertical the height of its post-impact jump does not exceed half the length of the thread, this periodic motion is orbitally stable, otherwise it is unstable. This result was obtained previously by Ivanov† using numerical calculations.

†IVANOV A. P., A qualitative theory of motion in systems with non-restoring couplings. Doctorate dissertation, Moscow, 14 November 1994.

8. NON-LINEAR OSCILLATIONS IN THE NEIGHBOURHOOD OF AN UNSTABLE TRAJECTORY

Suppose $\varkappa = 1/2 + \varepsilon$ $(0 < \varepsilon \ll 1)$. In this case the periodic motion (1.3) is orbitally unstable. We will fix the energy level corresponding to trajectories (1.3) and consider the nature of the non-linear oscillations of the particle m in its neighbourhood. We will again obtain the Hamilton function describing the oscillations using the corresponding simplectic mapping. Since the values of the parameter $\varkappa$ are assumed to be close to its boundary value, then, as in Section 7, we must investigate the mapping in the range of variation of v from O to 4π.

Assuming $q = \varepsilon^{1/2}X, p = \varepsilon^{1/2}Y$, we obtain from (4.2) and (4.3) the mapping $X_0, Y_0 \to X_2, Y_2$ in the form

$$\begin{aligned} X_2 &= (1+16\varepsilon)X_0 - 8\pi^{-1}(1+4\varepsilon)Y_0 + \varepsilon F(X_0, Y_0) + O(\varepsilon^{3/2}) \\ Y_2 &= -4\pi\varepsilon X_0 + (1+16\varepsilon)Y_0 + \varepsilon G(X_0, Y_0) + O(\varepsilon^{3/2}) \end{aligned} \tag{8.1}$$

where F and G are the functions from (7.1).

After two canonical changes of variables $X, Y \to X', Y'$ using the formulae

$$X = \sqrt{2}\pi^{-1}(1+2\varepsilon)X', \quad Y = \sqrt{2}\pi[2(1+2\varepsilon)]^{-1}Y'$$

and $X', Y' \to X^*, Y^*$, specified by the generating function

$$S = X'Y^*(1-(11/60)\varepsilon Y^{*2})$$

mapping (8.1) takes the following form

$$X_2^* = (1+16\varepsilon)X_0^* - 4\pi Y_0^* + \varepsilon F^*(X_0^*, Y_0^*) + O(\varepsilon^{3/2})$$

$$Y_2^* = -8\varepsilon\pi^{-1}X_0^* + (1+16\varepsilon)Y_0^* + \varepsilon G^*(X_0^*, Y_0^*) + O(\varepsilon^{3/2})$$

where F^* and G^* are the functions from (7.2). The following Hamilton function corresponds to this mapping

$$K = -\tfrac{1}{2}Y^{*2} + \varepsilon(\pi^{-2}X^{*2} + \tfrac{8}{3}Y^{*2}) - \varepsilon\tfrac{3}{4}\pi^{-4}X^{*4} + O(\varepsilon^{3/2}) \tag{8.2}$$

Quantities $O(\varepsilon^{3/2})$ in (8.2) are 4π-periodic in v.

The canonical equations with Hamiltonian (8.2) describe non-linear oscillations in the neighbourhood of the trajectory (1.3). If the mappings are not used, the change from the q, p variables to the X^*, Y^* variables can be obtained by a canonical transformation, 4π-periodic in v and analytic in q and p, which reduces Hamiltonian (3.1) to the form (8.2).

To facilitate further calculations we will make one more canonical change of variables $X^*, Y^* \to Q, P$, by putting

$$X^* = (\tfrac{2}{3})^{1/2}(1-\tfrac{8}{3}\varepsilon)\pi Q, \quad Y^* = (1-\tfrac{8}{3}\varepsilon)^{-1}P$$

and we will introduce the new independent variable $\zeta = (3/2)^{1/2}\pi^{-1}\upsilon$. In the new variables the motion is described by a Hamiltonian of the form

$$K = -\tfrac{1}{2}P^2 + \tfrac{1}{3}\varepsilon(2Q^2 - Q^4) + O(\varepsilon^{3/2}) \tag{8.3}$$

Omitting the term $O(\varepsilon^{3/2})$ in (8.3) we obtain an approximate ("unperturbed") system with Hamiltonian

$$K^{(0)} = -\tfrac{1}{2}P^2 + \tfrac{1}{3}\varepsilon(2Q^2 - Q^4) \tag{8.4}$$

The unperturbed system has the integral $K^{(0)} = c =$ const. The phase pattern is shown in Fig. 2. When $c > \varepsilon/3$ motion is impossible. If $c > \varepsilon/3$ the system is in one of the equilibrium positions $Q = \pm 1, P = 0$. In the phase plane these correspond to stable singular points—centres.

When $0 < c < \varepsilon/3$ (the range of oscillations) oscillations occur in the neighbourhood of positions of equilibrium. These oscillations can be described using Jacobi elliptic functions

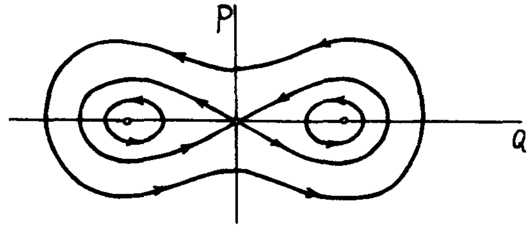

Fig. 2.

$$Q=\pm a\,\mathrm{dn}(\eta,k),\quad P=\pm\tfrac{2}{3}[6(\varepsilon-3c)]^{1/2}\,\mathrm{sn}(\eta,k)\,\mathrm{cn}(\eta,k)$$
$$k^2=(a^2-b^2)/a^2\quad(0<k<1),\quad \eta=(2\varepsilon/3)^{1/2}a(\zeta+\zeta_0)\tag{8.5}$$
$$a^2=1+(1-3c/\varepsilon)^{1/2},\quad b^2=1-(1-3c/\varepsilon)^{1/2}\quad(a>b>0)$$

Here and below ζ_0 is an arbitrary constant, and the upper and lower signs relate to trajectories in the right and left of the half-planes of the phase pattern, respectively.

The frequency (with respect to ζ) of the oscillations of the quantities q and p is specified by the equation

$$\omega=\tfrac{1}{3}\pi a(6\varepsilon)^{1/2}K^{-1}(k)\tag{8.6}$$

where $K(k)$ is the complete elliptic integral of the first kind. When $c\to\varepsilon/3$ we obtain from (8.6) the frequency of small oscillations in the neighbourhood of the equilibria $Q=\pm1, P=0$, equal to $2/3(6\varepsilon)^{1/2}$.

The value $c=0$ of the integral $K^{(0)}=\text{const}$ corresponds either to the unstable equilibrium $Q=P=0$ (a saddle in the phase plane), or doubly asymptotic homoclinic trajectories—separatrices. On these we have

$$Q=\pm\sqrt{2}\,\mathrm{ch}^{-1}\eta,\quad P=\pm\tfrac{2}{3}\sqrt{6\varepsilon}\,\mathrm{sh}\,\eta\,\mathrm{ch}^{-2}\eta$$
$$\eta=\tfrac{2}{3}\sqrt{3\varepsilon}(\zeta+\zeta_0)$$

When $c<0$ (the region of rotations) non-linear oscillations occur for which the phase trajectories in Fig. 2 envelope all three equilibrium positions. On these trajectories

$$Q=a\,\mathrm{cn}(\eta,k),\quad P=\tfrac{2}{3}a[9\varepsilon(\varepsilon-3c)]^{1/4}\,\mathrm{sn}(\eta,k)\,\mathrm{dn}(\eta,k)$$
$$k^2=a^2/(a^2+b^2)\quad(\sqrt{2}/2<k<1),\quad \eta=\tfrac{2}{3}[9\varepsilon(\varepsilon-3c)]^{1/4}(\zeta+\zeta_0)\tag{8.7}$$
$$a^2=1+(1-3c/\varepsilon)^{1/2},\quad b^2=-1+(1-3c/\varepsilon)^{1/2}\quad(a>b>0)$$

In the region of rotations the frequency is given by the expression

$$\omega=\tfrac{1}{3}\pi[9\varepsilon(\varepsilon-3c)]^{1/4}K^{-1}(k)\tag{8.8}$$

In the regions of oscillations and rotations the Hamilton function (8.4) can be reduced to action-angle variables I', w'. In these variables $K^{(0)}=c(I')$, where $\omega=\partial K^{(0)}/\partial I'$. The Hamiltonian $K^{(0)}$ satisfies the non-degeneracy condition $\partial^2K^{(0)}/\partial I'^2\neq0$. In fact, in the region of oscillations we obtain from (8.5) and (8.6) that

$$\frac{\partial\omega}{\partial I'}=\frac{\pi^2(2-k^2)}{2k^4(1-k^2)K^3}[(2-k^2)E-2(1-k^2)K]\tag{8.9}$$

where $E(k)$ is the complete elliptic integral of the second kind. Noting that the derivative with respect to k of the expression in square brackets in (8.9) is equal to $3k(K-E)$ and is positive for all k, and this expression is equal to zero when $k = 0$, we obtain that it is positive for all k from the interval (0, 1). Consequently, $\partial\omega/\partial I' > 0$ and the non-degeneracy condition is satisfied.

In the case of rotations we have from (8.7) and (8.8)

$$\frac{\partial\omega}{\partial I'} = -\frac{\pi^2(2k^2-1)}{8k^2(1-k^2)K^3}[(1-k^2)K + (2k^2-1)E] \tag{8.10}$$

Taking into account the fact that $1/2 < k^2 < 1$ in the region of rotations, we obtain from (8.10) that $\partial\omega/\partial I' < 0$, and the non-degeneracy condition is also satisfied here.

We will now consider a perturbed system with the complete Hamiltonian (8.3). The unstable equilibrium $Q = P = 0$ also exists in the perturbed system, and it corresponds to the periodic motion (1.3) of the particle m. Using the Poincaré theory of periodic motions [13] and Moser's theorem on invariant curves [10] it can be shown that an orbitally stable periodic motion of the particle m with a period equal to twice the period of the motion (1.3) can be generated from the stable equilibria $Q = \pm 1, P = 0$ of the unperturbed system.

It also follows from the non-degeneracy of the Hamiltonian $K^{(0)}$ and Moser's theorem on invariant curves that if the positive quantity ε is sufficiently small, the trajectories of the particle m which start fairly close to the unstable trajectory (1.3) always remain in its neighbourhood. We can obtain an estimate of the parameters of this neighbourhood by making use of the above analysis of an unperturbed system: if the trajectories of the unperturbed system with Hamiltonian (8.3) begin fairly close to the origin of coordinates $Q = P = 0$, then in further motion $|Q| < \sqrt{2}(1 + g_1)$, $|P| < 1/3\sqrt{(6\varepsilon)}(1 + g_2)$, where g_1 and g_2 can be as small as desired when ε and Q_0 and P_0 approach zero.

This research was carried out with financial support from the Russian Foundation for Basic Research (93-013-16257) and the International Science Foundation (MFG300).

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Translated by R.C.G.